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VARIATIONAL APPROACH OF A MAGNETIC SHAPING PROBLEM

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VARIATIONAL APPROACH OF A MAGNETIC SHAPING PROBLEM

APPROCHE VARIATIONNELLE D'UN PROBLEME DE FORMAGE ELECTROMAGNETIQUE

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Programme 2

Résumé : Nous étudions un problème de formage électromagnétique de métal à partir d'une fonctionnelle d'énergie où l'on néglige la tension superficielle. Nous obtenons l'existence d'un potentiel magnétique minimisant cette fonctionnelle, et montrons qu'il est uniformément lipschitzien.

Abstract : For a problem of electromagnetic casting in absence of surface tension, we make a study from the minimization of an energy functional. We prove existence and global Lipschitz-continuity of a magnetic potential which minimizes this functional.

Key words : Free boundary, magnetic casting.

AMS (MOS) subject classification : 35R35,35A15,49A22.

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1. Modelling.

In order to shape a tube, we have a device that we suppose to be cylindrical, vertical and infinite. Figure 1 represents a horizontal cross section of it. From the outside to the inside we have a fixed mould, then a layer of molten metal, next the vacuum and finally in the central part an inductor where a high frequency alternative current is running.

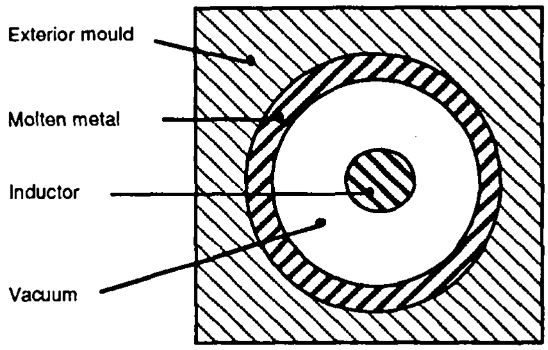

Figure 1

We assume that the conductivity of the molten metal is infinite. The surface tension on its boundary and more generally all forces in the device not produced by the electromagnetic field are neglected.We suppose we are in a perfectly cylindrical steady state. Moreover we assume that the Laplace forces acting on the inside boundary of the molten metal are strong enough to keep it away from the inductor and that the metal covers completely the wall of the exterior mould.

Let Ω be the (open) part of the horizontal plane which corresponds to the vacuum and the inductor. Its boundary $\partial\Omega$ is the interface between the vacuum and the molten metal. Let Ω^c denote the complementary set of Ω. Since we have a perfectly cylindrical steady state, the magnetic induction field $\vec{B}$ has to be horizontal and the running current vertical. The conductivity is infinite in the molten metal, so the induced current is of pellicular type, (i.e. supported by $\partial\Omega$), and the molten metal realises a perfect shield so that $\vec{B} = 0$ outside $\bar{\Omega}$. From the Maxwell equations we have

(1) $\quad \operatorname{div} \vec{B} = 0 \qquad$ in $\mathbb{R}^2$,

(2) $\quad \operatorname{curl} \vec{B} = \mu_0 \, j_0 \qquad$ in $\mathbb{R}^2$,

where μ_0 is the magnetic permeability of the vacuum and j_0 denotes the (vertical component of the) current density.

Writing the equilibrium between the Laplace forces and the magnetic pressure at the boundary $\partial\Omega$ of the molten metal, we get the relation

$$(3) \qquad \frac{1}{4\mu_0} |\vec{B}|^2 = \text{constant} \qquad \text{on } \partial\Omega,$$

see for example [SNEYD & MOFFAT, 1982].

From equation (1), we can find a magnetic potential φ such that $\vec{B} = \overrightarrow{\text{curl}}\, \varphi$ in $\mathbb{R}^2$. As φ is defined up to an arbitrary constant and $\vec{B} = 0$ in Ω^c, we can choose φ such that $\varphi = 0$ in Ω^c. We introduce now the functional

$$J(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} \vec{B}^{\,2}\, dx - \int_{\mathbb{R}^2} j\, \varphi\, dx\,,$$

where $j = \mu_0\, j_0$ in the inductor, $j = 0$ outside, which represents the total magnetic energy of the device. Since $|\vec{B}| = |\vec{\nabla}\varphi|$, we can write

$$(4) \qquad J(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} \vec{\nabla}\varphi^2\, dx - \int_{\mathbb{R}^2} j\, \varphi\, dx\,.$$

We assume that j_0 is given in the conductor (i.e. j is given) and that the quantity of molten metal is fixed, which means that the area of the horizontal section corresponding to the molten metal is fixed, or equivalently the area $|\Omega|$ of Ω is equal to a fixed value γ.

In a more general context, [HENROT & PIERRE,1 1989] have studied the inverse problem : for a given domain Ω, find a current density j such that (1), (2) and (3) hold ; see also [COULAUD and al.]. The direct problem : for a given current density j and a given positive number γ , find $\vec{B}$ and Ω such that (1), (2), (3) and $|\Omega| = \gamma$ has been studied by many authors, [S & M, 1982], [GAGNOUD & SERO GUILLAUME, 1986], [HENROT and al, 1989], [DESCLOUX, 1990, 1991]. In a more general setting, they start from equations (1), (2), (3) and remark that the solutions satisfy some stationary properties of the energy functional J. In [D, 1990] some stability concepts are introduced and it is shown that the interior shaping problem can have stable solutions. In fact these results give more insight to our approach. We look for a magnetic potential which minimizes J under the constraint $|\Omega| = \gamma$ where we set $\Omega = \{x \in \mathbb{R}^2 \,;\, \varphi(x) \neq 0\}$. Then we get easily an existence theorem for φ and if φ and Ω are smooth enough, we prove that $\vec{B} = \overrightarrow{\text{curl}}\, \varphi$ satisfies (1), (2), (3). The major difficulty we have to deal is the one of the regularity. We have only partially succeeded in our attempt to get such results. We show that φ is uniformly Lipschitz continuous in $\mathbb{R}^2$ and that Ω is an open set. In fact, at least for γ large enough, our guess is that the boundary $\partial\Omega$ and the function φ in the vacuum up to $\partial\Omega$ are analytic.

2. Existence theorem.

Let K be a compact subset of $\mathbb{R}^2$ such that its complementary set K^c is a connected set, $j \in L^\infty(\mathbb{R}^2)$ be a function such that

(5) $\quad$ support $(j) \subset K$,

and let γ be a real number such that

(6) $\quad \gamma > \gamma_0 = \text{area } (K).$

The Sobolev space $H^m(\mathbb{R}^2)$ denotes the set of functions whose partial derivatives up to the order m are in $L^2(\mathbb{R}^2)$.
For a function v in $H^1(\mathbb{R}^2)$ we set $\Omega_v = \{x \in \mathbb{R}^2 ; v(x) \neq 0\}$, $\chi_v(x)$ the function defined by $\chi_v(x) = 1$ if $v(x) \neq 0$, $\chi_v(x) = 0$ if $v(x) = 0$, and $|\Omega_v| = \|\chi_v\| = \int_{\mathbb{R}^2} \chi_v(x)\, dx$. We define the two sets

$$B(\gamma) = \{w \in H^1(\mathbb{R}^2) ; \|\chi_w\| = \gamma\},$$

$$A(\gamma) = \{w \in H^1(\mathbb{R}^2) ; \|\chi_w\| \leq \gamma\}.$$

With the functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^2} \vec{\nabla} v^{\,2}\, dx - \int_{\mathbb{R}^2} j\, v\, dx ,$$

we associate the two problems

$$(\mathrm{P}_0) \qquad \begin{cases} \text{Find } u \in B(\gamma) \text{ such that} \\ \forall v \in B(\gamma) , \quad J(u) \leq J(v) \end{cases}$$

and

$$(\mathrm{P}) \qquad \begin{cases} \text{Find } u \in A(\gamma) \text{ such that} \\ \forall v \in A(\gamma) , \quad J(u) \leq J(v) . \end{cases}$$

From our modelling, we are concerned in the problem (P_0) but, from a mathematical point of view, the problem (P) is easier to study . Fortunately we have the relation

Lemma 1. *We assume that the situation described in* (7) *does not hold*
(7) $\quad$ *there exists* $u \in H^1(\mathbb{R}^2)$ *with* $-\Delta u = j$ *in* $\mathbb{R}^2$ *and* $\|\chi_u\| < +\infty$;
then the problems (P_0) *and* (P) *are equivalent.*

Proof. From the density of $B(\gamma)$ in $A(\gamma)$, u solution of (P_0) implies u solution of (P). Conversely let u be a solution of (P) and $\gamma' = \|\chi_u\|$. If u were not a solution of (P_0), we should have $\gamma' < \gamma$ and then

$$\forall \varphi \in H^1(\mathbb{R}^2) \text{ with } \|\chi_\varphi\| \leq \gamma - \gamma' \quad , \qquad J(u) \leq J(u+\varphi)$$

whence $\displaystyle\int_{\mathbb{R}^2} \vec{\nabla} u\, \vec{\nabla}\varphi\, dx = \int_{\mathbb{R}^2} j\, \varphi\, dx$,

so that we can conclude that (7) holds. ■

Remarks. 1) When the situation (7) occurs we have, $\forall v \in H^1(\mathbb{R}^2)$, $J(u) \leq J(v)$, moreover u is harmonic in K^c . With $\|\chi_u\| < +\infty$ that induces $u = 0$ in K^c and therefore $|\Omega_u| \leq \gamma_0$. For the physical model such a j is not suitable since we must have no contact between molten metal and the inductor.

2) When $\int_{\mathbb{R}^2} j\, dx \neq 0$, the situation (7) cannot occur.

Theorem 2. *Problem* (P) *admits at least one solution.*

Proof. We first remark that, from the Schwarz symmetrization principle, we have [SPERNER, 1974]

$$\forall v \in A(\gamma)\,, \quad \| v \|_{L^2(\mathbb{R}^2)} \leq C\, \gamma^{1/2}\, \| \vec{\nabla} v \|_{L^2(\mathbb{R}^2)} \ ,$$

therefore $J(v) \geq -\frac{1}{2}\, C^2\, \gamma\, \| j \|_{L^2(\mathbb{R}^2)}$, and $\inf\limits_{v \in A(\gamma)} J(v) > -\infty$. On the other hand, $A(\gamma)$ is weakly closed in $H^1(\mathbb{R}^2)$. Indeed if $v_n \in A(\gamma) \to v$ weakly in $H^1(\mathbb{R}^2)$, eventually using a subsequence, we can assume $v_n \to v$ a.e. in $\mathbb{R}^2$. Then let F be a compact subset of $\mathbb{R}^2$, since $\|\chi_{v_n}\| \leq \gamma$, we have

$$\int_F \frac{|v|}{\sqrt{v^2+\varepsilon^2}}\, dx = \lim_{n\to\infty} \int_F \frac{|v_n|}{\sqrt{v_n^2+\varepsilon^2}}\, dx \leq \gamma \,.$$

Therefore, taking the limit as $\varepsilon \to 0$, $\int_F \chi_v\, dx \leq \gamma$. Since this inequality holds for all compact F in $\mathbb{R}^2$, we deduce $v \in A(\gamma)$. Then the theorem follows from the weak lower semi-continuity of J. ■

Remark. A similar existence result may be found in [PIRONNEAU, 1984].

Hereafter u denotes a solution of (P). We shall see in theorem 3 that u is continuous, so that $\Omega = \Omega_u = \{x \,;\, u(x) \neq 0\}$ is an open set. Since for all φ supported by Ω, we have $J(u) \leq J(u+\varphi)$, so

(8) $\qquad -\Delta u = j \qquad \text{in } \Omega$;

and by the definition $u = 0$ in Ω^c.

3. A regularity result.

Let us introduce now the notations

$$\text{(9)} \qquad \partial v = \frac{1}{2}\left(\frac{\partial v}{\partial x_1} - i\, \frac{\partial v}{\partial x_2}\right)\ ,\ \bar{\partial} v = \frac{1}{2}\left(\frac{\partial v}{\partial x_1} + i\, \frac{\partial v}{\partial x_2}\right)\ ,\ y = y_1 + i\, y_2\,,$$

(but $dy = dy_1\, dy_2$ is the Lebesgue measure in $\mathbb{R}^2$). Let $\mathcal{D}(\mathbb{R}^2)$ denote the set of compact supported functions which are infinitely differentiable in $\mathbb{R}^2$. When there is no ambiguity, we shall write

$\int -$ and $\|.\|_{L^p}$ in place of $\int_{\mathbb{R}^2} -$ and $\|.\|_{L^p(\mathbb{R}^2)}$.

The main result of this section is

Theorem 3. *Let u be a solution of* (P). *Then there exists a positive number λ such that*

$$\text{(10)} \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^2)\,,\ 4\int (\partial u)^2\, (\bar{\partial}\varphi)\, dx = 2\int j\, \partial u\, \varphi\, dx + \lambda \int \chi_u \partial\varphi\, dx\,.$$

Moreover we have

$$\text{(11)} \qquad (\partial u)^2\,(x) = -\frac{1}{2\pi}\int (j\,\partial u)(x-y) \frac{1}{\bar{y}}\, dy - \frac{\lambda}{4\pi}\int \chi_{u(x-y)}\ \frac{1}{\bar{y}^2}\, dy,$$

and the function u is uniformly Lipschitz continuous in $\mathbb{R}^2$.

Proof. a) Let $\vec{\theta} \in (C^\infty(\mathbb{R}^2))^2$ be such that its derivative $D\vec{\theta} \in (L^\infty(\mathbb{R}^2))^4$ and $\int \chi_u \operatorname{div} \vec{\theta}\, dx > 0$. For $\delta > 0$, we consider $F_\delta(x) = x + \delta\, \theta(x)$ and $u_\delta = u \circ F_\delta$. Clearly if δ is small enough, F_δ is a C^∞ diffeomorphism, furthermore $u_\delta \in H^1(\mathbb{R}^2)$ and

$$\int \chi_{u_\delta}(x)\, dx = \int \chi_u(y)\, dF_\delta^{-1}(y) = \int \chi_u(y)\, dy - \delta \int \chi_u(y)\ \operatorname{div}\vec{\theta}(y)\, dy + O(\delta^2),$$

so that for $\delta > 0$ small enough, $u_\delta \in A(\gamma)$. Writing $\frac{1}{\delta}\,(J(u_\delta) - J(u)) \geq 0$ and taking the limit as $\delta \to 0_+$ we obtain

$$\text{(12)} \qquad \int \{D\vec{\theta}(\vec{\nabla} u,\vec{\nabla} u) - \tfrac{1}{2}\operatorname{div}\vec{\theta}\ \vec{\nabla} u^2\}\, dx - \int\ j\, \vec{\nabla} u.\ \vec{\theta}\, dx\ \geq\ 0.$$

We consider now a function $\vec{\theta}$ such that $\int \chi_u \operatorname{div}\vec{\theta}\, dx = 0$. We still obtain that (12) holds by using $\vec{\theta_\varepsilon}(x) = \theta(x) + \varepsilon\, x$ and then taking the limit as $\varepsilon \to 0_+$, but now changing $\vec{\theta}$ in $-\vec{\theta}$, we have the equality in (12).

For an arbitrary function $\vec{\theta}$, we set $\vec{\theta_1} = \vec{\theta} - \frac{1}{2\gamma}\int \chi_u \operatorname{div}\vec{\theta}\, dx)\, \vec{x}$, so that $\int \chi_u \operatorname{div}\vec{\theta_1}\, dx = 0$. The equality in (12) for $\vec{\theta_1}$ leads

(13) $$\int \{D\vec{\theta}(\nabla\vec{u},\nabla\vec{u}) - \tfrac{1}{2}\operatorname{div}\vec{\theta}\ \nabla\vec{u}^2\}\, dx - \int j\,\nabla\vec{u}.\ \vec{\theta}\, dx = \lambda \int \chi_u \operatorname{div}\vec{\theta}\, dx,$$

with

(14) $$\lambda = -\frac{1}{\gamma}\int j\,\nabla\vec{u}\ .\ \vec{x}\, dx.$$

We remark that λ is non negative when using (12) with $\vec{\theta} = \vec{x}$.

b) Equation (10) is exactly (13) rewritten with the notations (9) and $\varphi = \theta_1 - i\,\theta_2$. Using that $\int \bar{\partial}\varphi(y)\ (\bar{y})^{-1}\ dy = -\pi\,\varphi(0)$ and $\int \bar{\partial}\varphi(y)\ (\bar{y})^{-2}\ dy = -\pi\,\partial\varphi(0)$, and setting

(15) $$a(x) = -\frac{1}{2\pi}\int (j\,\partial u)(x-y)\frac{1}{\bar{y}}\, dy - \frac{\lambda}{4\pi}\int \chi_u(x-y)\frac{1}{\bar{y}^2}\, dy\ ,$$

we deduce from (10) that

$$\forall\varphi \in \mathcal{D}(\mathbb{R}^2)\ ,\quad \int ((\partial u)^2 - a)\,\bar{\partial}\varphi\, dx = 0.$$

Therefore $(\partial u)^2 - a$ is harmonic in $\mathbb{R}^2$.

We know that $u \in H^1(\mathbb{R}^2)$, thus $(\partial u)^2 \in L^1(\mathbb{R}^2)$. Since, $j\,\partial u \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $y \to (\bar{y})^{-1} \in L^p(\mathbb{R}^2) + L^q(\mathbb{R}^2)$ for all $1 \le q < 2$ and $2 < p \le +\infty$, the function

(16) $$s(x) = -\frac{1}{2\pi}\int (j\,\partial u)(x-y)\ (\bar{y})^{-1}\ dy$$

satisfies $s \in L^p$ for all $2 < p < +\infty$. We introduce now the Beurling transform T given by

(17) $$Tg(x) = \frac{1}{4\pi}\int g(x-y)\ (\bar{y})^{-2}\ dy\ ,$$

it acts from $L^p(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ for all $1 < p < +\infty$, (see for instance [ZYGMUND, 1971]). As $\chi_u \in L^p(\mathbb{R}^2)$ for all p and $a = s - \lambda\, T\,\chi_u$, we have $a \in L^p(\mathbb{R}^2)$ for all $2 < p < +\infty$. Since the harmonic function $(\partial u)^2 -$ a belongs to $L^1(\mathbb{R}^2) + L^p(\mathbb{R}^2)$, we have $(\partial u)^2 - a = 0$ in $\mathbb{R}^2$, therefore

$$(\partial u)^2 = a \in L^p(\mathbb{R}^2) \text{ for all } 2 < p < +\infty$$

and (11) is proven.

c) Now we have $j\,\partial u \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for all $1 \le p < +\infty$, therefore $s \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Let us introduce

(18) $$T_\varepsilon(x) = \frac{1}{4\pi} \int_{|y|\geq\varepsilon} \chi_u(x-y)\ (\bar{y})^{-2}\ dy\ .$$

We admit for a while the result

Lemma 4. *There exists a constant C such that*

(19) $$\forall x \in \mathbb{R}^2,\ \forall \varepsilon > 0,\ |T_\varepsilon(x)| \leq C + \|T\chi_u\|_{L^\infty(\Omega^c)}\ .$$

Since $\lim\limits_{\varepsilon\to 0} T_\varepsilon = T\chi_u$ a.e. in $\mathbb{R}^2$ and $\partial u = 0$ in Ω^c, we have $\lambda\ T\chi_u = s$ in Ω^c and $\lambda\, T\chi_u \in L^\infty(\mathbb{R}^2)$. We deduce from (11) that $\partial u \in L^\infty(\mathbb{R}^2)$ and

$$\|\partial u\|^2_{L^\infty(\mathbb{R}^2)} \leq 2\ \|\ s\ \|_{L^\infty(\mathbb{R}^2)} + C\,\lambda\ .$$

From (14) we get $\lambda \leq \gamma^{-1}\ \|\ j\ \vec{\nabla} u.\vec{x}\ \|_{L^1} \leq \|\ jx\ \|_{L^1}\ \|\partial u\|_{L^\infty}$

and from (16) $\|\ s\ \|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}}\ (\|\ j\ \|_{L^\infty} + \|\ j\ \|_{L^1})\ \|\partial u\|_{L^\infty}\ .$

Therefore we have

(20) $$\|\partial u\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2\pi}}\ (\|\ j\ \|_{L^1} + \|\ j\ \|_{L^\infty}) + C\ \|\ jx\ \|_{L^\infty}.$$

Note that this estimate does not depend on γ.

d) To be complete, we have to prove Lemma 4. Without loss of generality we can suppose $x = 0$. Let $\varphi(\varepsilon) = T_\varepsilon(0)$, $D_\varepsilon = \{x \in \mathbb{R}^2\ ;\ \frac{\varepsilon}{2} < |x| \leq \varepsilon\}$, $B_\varepsilon = \{x \in \mathbb{R}^2\ ;\ |x| < \varepsilon\}$, $\alpha(\varepsilon) = \frac{\text{meas}(D_\varepsilon \cap \Omega^c)}{\text{meas}(D_\varepsilon)}$ and $C_0 = \|T\chi_u\|_{L^\infty(\Omega^c)}$. We have

$$\varphi(\varepsilon) - \varphi(\tfrac{\varepsilon}{2}) = \frac{1}{4\pi} \int_{D_\varepsilon} (1-\chi_u(y))\ (\bar{y})^{-2}\, dy = \frac{1}{4\pi} \int_{D_\varepsilon\cap\Omega^c} (\bar{y})^{-2}\ dy,$$

thus

(21) $$|\varphi(\varepsilon) - \varphi(\tfrac{\varepsilon}{2})| \leq \frac{3}{4}\ \alpha(\varepsilon).$$

On the other hand we set

$$f_1(x) = \chi_u(x) - 1\ ,\quad f_0(x) = 1\ ,\quad f_2(x) = 0\ ,\qquad \text{if } |x| \leq \varepsilon\ ,$$

and $$f_1(x) = 0\ ,\quad f_0(x) = 0\ ,\quad f_2(x) = \chi_u(x)\ ,\qquad \text{if } |x| > \varepsilon\ ,$$

so that $$\varphi(\varepsilon) = (Tf_2)\,(0)\ ,\qquad \chi_u = f_1 + f_2 + f_0\ .$$

We can write $$\varphi(\varepsilon) = T\chi_u(x) - Tf_1(x) - Tf_0(x) + (Tf_2(0) - Tf_2(x)).$$

We have $Tf_0(x) = 0$ for $|x| < \varepsilon$ (and $= -\varepsilon^2/(4\bar{x}^2)$ for $|x| > \varepsilon$), some easy computations give the (non optimal) estimate,

$$|Tf_2(0) - Tf_2(x)| \leq 2, \qquad \text{for } |x| \leq \frac{\varepsilon}{2} \ ,$$

therefore we have

$$|\varphi(\varepsilon)| \leq C_0 + |Tf_1(x)| + 2, \qquad \text{a.e. in } B_{\varepsilon/2} \cap \Omega^c.$$

From the continuity of T from $L^1(\mathbb{R}^2)$ into weak $L^1(\mathbb{R}^2)$, see for instance [COIFMAN & MEYER, 1978], there exists C_1 such that

$$\forall \mu > 0 \ , \qquad \mu \ \mathrm{meas}\{x \in \mathbb{R}^2, |Tf_1(x)| \geq \mu\} \leq C_1 \ \|f_1\|_{L^1} = C_1 \ \mathrm{meas}\,(B_\varepsilon \cap \Omega^c).$$

With the choice

$$\mu = |\varphi(\varepsilon)| - C_0 - 2, \text{ (when such a } \mu \text{ is } > 0),$$

that gives $\quad (|\varphi(\varepsilon)| - C_0 - 2)\ \mathrm{meas}\,(B_{\varepsilon/2} \cap \Omega^c) \leq C_1 \ \mathrm{meas}\,(B_\varepsilon \cap \Omega^c)$,

whence $|\varphi(\varepsilon)| \leq C_0 + 2 + C_1 \dfrac{\mathrm{meas}\,(B_\varepsilon \cap \Omega^c)}{\mathrm{meas}\,(B_{\varepsilon/2} \cap \Omega^c)} \leq C_0 + 2 + C_1 + C_1 \dfrac{\mathrm{meas}\,(D_\varepsilon \cap \Omega^c)}{\mathrm{meas}\,(D_{\varepsilon/2} \cap \Omega^c)}$,

therefore we have, (if $\alpha(\varepsilon/2) \neq 0$),

$$(22) \qquad |\varphi(\varepsilon)| \leq C_0 + 2 + C_1 + 4C_1 \frac{\alpha(\varepsilon)}{\alpha(\varepsilon/2)} \ .$$

If $\alpha(\varepsilon) < 2\ \alpha(\varepsilon/2)$ then we have $|\varphi(\varepsilon)| \leq C_0 + 9\ C_1 + 2$. Otherwise there exists an integer k such that

$$\alpha(2^j\varepsilon) \leq 2^{-1}\ \alpha(2^{j+1}\varepsilon)\ , \text{ for } j = 1, \dots, k{-}1 \qquad \text{and} \quad \alpha(2^k\varepsilon) > 2^{-1}\ \alpha(2^{k+1}\varepsilon).$$

Then we have $\qquad |\varphi(2^{k+1}\varepsilon)| \leq C_0 + 9\ C_1 + 2$ and

$$\alpha(2^j\varepsilon) \leq 2^{j-k}\ \alpha(2^k\varepsilon) \leq 2^{j-k} \qquad \text{for } j = 1, \dots, k.$$

From (21) we get

$$|\varphi(\varepsilon) - \varphi(\tfrac{\varepsilon}{2})| \leq \frac{3}{4}\ (\alpha(2^{k+1}\varepsilon) + \alpha(2^k\varepsilon) + \dots + \alpha(2\varepsilon)) \leq 9/4$$

so that $\qquad |\varphi(\varepsilon)| \leq C_0 + 9\ C_1 + 17/4$,

and (19) holds with $C = 9\ C_1 + 17/4$. ■

Remark. We cannot have $\lambda = 0$ in (11). Indeed $\lambda = 0$ would imply $(\partial u)^2$ is harmonic in K^c, so that $\partial u = 0$ in K^c whence $u = 0$ in K^c, which would be inconsistent with (6).

4. Conclusions.

We have got that the solution u of our minimization problem satisfies $-\Delta u = j$ in Ω. If the smoothness of Ω and u allows us to use the Green formula, we deduce from (10) that $4(\partial u)^2 = \lambda(v_1 + iv_2)^2$ on $\partial\Omega$, which means $\overrightarrow{\nabla u} = \pm\sqrt{\lambda}\ \vec{v}$; (here $\vec{v}$ is the unit vector normal to $\partial\Omega$). So that we get the equilibrium condition (3). With some stonger smoothness assumptions, it follows from the work of [H & P, 1989], see also [D, 1990], that u and $\partial\Omega$ have some analytic properties.

We assume now that j is fixed and satisfies $\int_\Omega j\,dx \neq 0$ (say $\int_\Omega j\,dx > 0$), that is the total current running in the inductor does not vanish. Let u_γ be the solution of (P), here γ appears to emphasize the dependence on γ, it is easy to show that $\lim\limits_{\gamma\to\infty} J(u_\gamma) = -\infty$; we have $J(u_\gamma) \geq -\int_{\mathbb{R}^2} j\, u_\gamma\, dx = -\int_K j\, u_\gamma\, dx$, and from (20),$\forall x, y \in \mathbb{R}^2, |u_\gamma(x) - u_\gamma(y)| \leq c\ |x-y|$. This implies $\lim\limits_{\gamma\to\infty}\ \min\limits_{x\in K} \{u_\gamma(x)\} = +\infty$, therefore, for γ large enough, the set K is included in Ω which means for γ large enough the molten metal has no contact with the inductor. We can also show that u_γ is non negative in $\mathbb{R}^2$.

We have proven that u is Lipschitz continuous, that gives very poor information on the smoothness on $\partial\Omega$. Note that the boundedness of ∂u implies by the lemma 4 that $|T_\varepsilon(x)|$ is uniformly bounded by a constant. Using the definition of T_ε and polar coordinates, we can easily see that $\partial\Omega$ cannot have a singularity of corner type, (except if the angle is 0, π or 2π , a cusp point is allowed).

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